

# MINIMIZING RATIONAL FUNCTIONS BY EXACT JACOBIAN SDP RELAXATION APPLICABLE TO FINITE SINGULARITIES

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**ABSTRACT.** This paper considers the optimization problem of minimizing a rational function. We reformulate this problem as polynomial optimization by the technique of homogenization. These two problems are shown to be equivalent under some generic conditions. The exact Jacobian SDP relaxation method proposed by Nie is used to solve the resulting polynomial optimization. We also prove that the assumption of nonsingularity in Nie's method can be weakened as the finiteness of singularities. Some numerical examples are given to illustrate the efficiency of our method.

## 1. INTRODUCTION

Consider the problem of minimizing a rational function

$$(1) \quad \begin{cases} r^* := \min_{x \in \mathbb{R}^n} \frac{p(x)}{q(x)} \\ \text{s.t. } h_1(x) = \cdots = h_{m_1}(x) = 0, \\ g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0. \end{cases}$$

where  $p(x), q(x), h_i(x), g_j(x) \in \mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_n]$ . As a special case, when  $\deg(q) = 0$ , (1) becomes a multivariate polynomial optimization which is NP-hard even when  $p(x)$  is a nonconvex quadratic polynomial and  $h_i(x)$ 's,  $g_j(x)$ 's are linear [21].

Some approaches using sum-of-squares relaxation to solve the minimization of (1) are proposed in [10, 18] and the core idea therein is in the following. Let  $S$  be the feasible set of (1). Suppose that  $r^* > -\infty$  and  $q(x)$  is nonnegative on  $S$  (otherwise replace  $\frac{p(x)}{q(x)}$  by  $\frac{p(x)q(x)}{q^2(x)}$ ), then  $\gamma \in \mathbb{R}$  is a lower bound of  $r^*$  if and only if  $p(x) - \gamma q(x) \geq 0$  on  $S$ . Thus the problem (1) can be reformulated as maximizing  $\gamma$  such that  $p(x) - \gamma q(x)$  is nonnegative on  $S$ , which is related to the representation of a nonnegative polynomial on a feasible set defined by several polynomial equalities and inequalities. As is well-known, a univariate polynomial is nonnegative on  $\mathbb{R}$  if and only if it can be represented as a sum-of-squares of polynomials (SOS) [25] which can be efficiently determined by solving a semidefinite program (SDP) [22, 23]. However, when  $n > 1$ , due to the fact that a nonnegative multivariate polynomial might not be an SOS [25], the problem (1) becomes very hard even if there are no constraints. Denote by  $M(S)$  the set of polynomials which can be written as

$$\sum_{i=1}^{m_1} \varphi_i(x) h_i(x) + \sum_{j=0}^{m_2} \sigma_j(x) g_j(x)$$

where  $\varphi_i(x) \in \mathbb{R}[x]$ ,  $g_0(x) = 1$  and  $\sigma_j(x)$ 's are SOS.  $M(S)$  is called the *quadratic module* generated by the defining polynomials of  $S$ . If  $M(S)$  is *archimedean*, which

means that there exists  $R > 0$  such that  $R - \|x\|_2^2 \in M(S)$ , then Putinar's Positivstellensatz [24] states that if a polynomial  $f(x)$  is positive on  $S$ , then it belongs to  $M(S)$ . If  $S$  is compact, then Schmüdgen's Positivstellensatz [26] states that if a polynomial  $f(x)$  is positive on  $S$ , then it can be represented as

$$(2) \quad f(x) = \sum_{i=1}^{m_1} \varphi_i(x) h_i(x) + \sum_{\nu \in \{0,1\}^{m_2}} \sigma_\nu(x) g_\nu(x),$$

where  $\varphi_i(x) \in \mathbb{R}[x]$ ,  $g_\nu(x) = g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$  and  $\sigma_\nu(x)$ 's are SOS. The set of polynomials which have representation (2) is called *preordering* which we denote by  $P(S)$ . Hence, if  $S$  in (1) is archimedean or compact, we can apply Putinar's Positivstellensatz or Schmüdgen's Positivstellensatz to maximize  $\gamma$  such that  $p(x) - \gamma q(x)$  belongs to  $M(S)$  or  $P(S)$ .

In this paper, we present a different way to obtain the minimum  $r^*$ . Given a polynomial  $f \in \mathbb{R}[x]$ , let  $\tilde{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$  and  $f^{hom}$  be the homogenization of  $f$ , i.e.  $f^{hom}(\tilde{x}) = x_0^{\deg(f)} f(x/x_0)$ . We reformulate the minimization of (1) by the technique of homogenization as the following polynomial optimization

$$(3) \quad \begin{cases} s^* := \min_{\tilde{x} \in \mathbb{R}^{n+1}} \tilde{p}(\tilde{x}) \\ \text{s.t. } h_1^{hom}(\tilde{x}) = \cdots = h_{m_1}^{hom}(\tilde{x}) = 0, \tilde{q}(\tilde{x}) = 1, \\ g_1^{hom}(\tilde{x}) \geq 0, \dots, g_{m_2}^{hom}(\tilde{x}) \geq 0, x_0 \geq 0, \end{cases}$$

where  $\tilde{p}(\tilde{x}) := x_0^{\max\{\deg(p), \deg(q)\}} p(x/x_0)$  and  $\tilde{q}(\tilde{x}) := x_0^{\max\{\deg(p), \deg(q)\}} q(x/x_0)$ . We show that these two problems are equivalent under some generic conditions. As a special case, they are always equivalent if there are no constraints in (1). The relations between the achievabilities of  $r^*$  and  $s^*$  are discussed.

Therefore, the problem of solving (1) becomes to efficiently solving problem (3). For general polynomial optimization, there has been much work on computing the infimum of the objective via SOS relaxations, see the survey [14] by Laurent and the references therein. A standard approach for solving polynomial optimizations is the hierarchy of semidefinite programming relaxations proposed by Lasserre [13]. Recently, under the assumption that the optimum is achievable, some gradient type SOS relaxations are presented in [3, 19]. When the optimum is an asymptotic value, we refer to the approaches proposed in [5, 6, 27, 28, 29]. However, to the best knowledge of the authors, the finite convergence of the above methods is unknown which means that we need to solve a big number of SDPs until the convergence is met. More recently, Nie [15] proposed a new type SDP relaxation using the minors of the Jacobian of the objective and constraints. It is shown [15] that the Jacobian SDP relaxation is exact under some generic assumptions. Therefore, in this paper we employ the Jacobian SDP relaxation to solve (3).

For another contribution of this paper, we prove that the assumptions under which the Jacobian SDP relaxation [15] is exact can be weakened. Let  $J$  be the set containing polynomials in the equality constraints and an arbitrary subset of the inequality constraints. In order to prove the finite convergence of the Jacobian SDP relaxation, it is assumed in [15] that the Jacobian of polynomials in  $J$  has full rank at any point in  $V(J)$  which is the variety defined by  $J$ . In other words, if the ideal  $\langle J \rangle$  generated by polynomials in  $J$  is radical and its codimension equals the number of these polynomials, the variety  $V(J)$  needs to be nonsingular to guarantee the finite convergence. In this paper, we prove that the nonsingularity in the assumptions

can be replaced by the finiteness of singularities. More specifically, we show that if there are only finite points in  $V(J)$  such that the Jacobian of polynomials in  $J$  is a rank deficient matrix, then the Jacobian SDP relaxation [15] is still exact. We also give an example to illustrate the correctness of our result.

Another possible and natural reformulation of (1) is

$$(4) \quad \begin{cases} \bar{s}^* := \min_{x \in \mathbb{R}^n, y \in \mathbb{R}} p(x)y \\ \text{s.t. } h_1(x) = \dots = h_{m_1}(x) = 0, \quad q(x)y = 1, \\ g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0. \end{cases}$$

Clearly, if  $r^*$  is achievable in (1), then (4) is equivalent to (1) and we always have  $r^* = \bar{s}^*$ . One might ask why we do not solve (4) instead of (3). The reason is that when we employ Jacobian SDP relaxation [15] to solve (3) or (4), we need to assume that the optimum is achievable. Actually,  $s^*$  in (3) is more likely to be achievable than  $\bar{s}^*$  in (4). To see this, note that when  $r^*$  is not achievable,  $\bar{s}^*$  can not be reached either. However,  $s^*$  might still be achievable when  $r^*$  is not. Some sufficient conditions are given in Theorem 2.7 and they are not necessary (see Example 4.6 and 4.9). For a simple example, consider the problem

$$\min_{x_1 \in \mathbb{R}} \frac{1}{x_1^2 + 1}.$$

Obviously,  $r^* = \bar{s}^* = 0$  and they are not achievable. However, we can reformulate it as

$$\begin{cases} s^* := \min_{x_0, x_1 \in \mathbb{R}} x_0^2 \\ \text{s.t. } x_1^2 + x_0^2 = 1. \end{cases}$$

Then  $s^* = 0$  and we have two minimizers  $(0, \pm 1)$  which verify that  $r^*$  is not achievable by (c) in Theorem 2.7.

This paper is organized as follows. In Section 2, we reformulate (1) as (3) by the technique of homogenization and investigate the relations between the achievabilities of the optima of these two optimizations. We introduce the Jacobian SDP relaxation [15] in Section 3 and show that the assumptions therein under which the Jacobian SDP relaxation is exact can be weakened. In Section 4, we first return to solving the problem (1) and make some discussions, then we give some numerical examples to illustrate the efficiency of our method.

**Notation.** The symbol  $\mathbb{N}$  (resp.,  $\mathbb{R}$ ,  $\mathbb{C}$ ) denotes the set of nonnegative integers (resp., real numbers, complex numbers). For any  $t \in \mathbb{R}$ ,  $\lceil t \rceil$  denotes the smallest integer not smaller than  $t$ . For integer  $n > 0$ ,  $[n]$  denotes the set  $\{1, \dots, n\}$  and for a subset  $J$  of  $[n]$ ,  $|J|$  denotes its cardinality. For  $x \in \mathbb{R}^n$ ,  $x_i$  denotes the  $i$ -th component of  $x$ . The symbol  $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$  (resp.,  $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ ) denotes the ring of polynomials in  $(x_1, \dots, x_n)$  with real (resp. complex) coefficients. For  $\alpha \in \mathbb{N}^n$ , denote  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . For  $x \in \mathbb{R}^n$  and  $\alpha \in \mathbb{N}^n$ ,  $x^\alpha$  denotes  $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . For a symmetric matrix  $X$ ,  $X \succeq 0$  (resp.,  $X \succ 0$ ) means  $X$  is positive semidefinite (resp., positive definite). For  $u \in \mathbb{R}^n$ ,  $\|u\|_2$  denotes the standard Euclidean norm.  $\mathcal{C}^k$  denotes the class of functions whose  $k$ -th derivatives are continuous.

## 2. MINIMIZING RATIONAL FUNCTIONS BY HOMOGENIZATION

In this section, we first reformulate the minimization of (1) as polynomial optimization (3) by the technique of homogenization and investigate the relations between the achievabilities of the optima of these two problems. Then we show that the condition under which the problems (1) and (3) are equivalent is generic.

**2.1. Reformulating the minimization of rational functions by homogenization.** Given a polynomial  $f \in \mathbb{R}[x]$ , let  $\tilde{x} = (x_0, x) = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$  and  $f^{hom}(\tilde{x})$  be the homogenization of  $f$ , i.e.  $f^{hom}(\tilde{x}) = x_0^{\deg(f)} f(x/x_0)$ . We define the following sets:

$$(5) \quad \begin{aligned} S &:= \{x \in \mathbb{R}^n \mid h_i(x) = 0, g_j(x) \geq 0, i \in [m_1], j \in [m_2]\}, \\ \tilde{S}_0 &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid h_i^{hom}(\tilde{x}) = 0, g_j^{hom}(\tilde{x}) \geq 0, x_0 > 0, i \in [m_1], j \in [m_2]\}, \\ \tilde{S} &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid h_i^{hom}(\tilde{x}) = 0, g_j^{hom}(\tilde{x}) \geq 0, x_0 \geq 0, i \in [m_1], j \in [m_2]\}. \end{aligned}$$

Recall that for integer  $n > 0$ ,  $[n]$  denotes the set  $\{1, \dots, n\}$ . Let  $\text{closure}(\tilde{S}_0)$  be the closure of  $\tilde{S}_0$  in  $\mathbb{R}^{n+1}$ . From the above definition, we immediately have

**Proposition 2.1.**  $f(x) \geq 0$  on  $S$  if and only if  $f^{hom}(\tilde{x}) \geq 0$  on  $\text{closure}(\tilde{S}_0)$ .

*Proof.* We first prove the “if” part. Suppose  $f^{hom}(\tilde{x}) \geq 0$  on  $\text{closure}(\tilde{S}_0)$ . If there exists a point  $u \in S$  such that  $f(u) < 0$ , then  $(1, u) \in \tilde{S}_0$ . Thus  $f^{hom}(1, u) = f(u) < 0$  which is a contradiction.

Next we prove the “only if” part. Suppose  $f(x) \geq 0$  on  $S$  and consider a point  $(u_0, u) \in \mathbb{R}^{n+1}$  in the  $\text{closure}(\tilde{S}_0)$ . There exists a sequence  $\{(u_{k,0}, u_k)\} \in \tilde{S}_0$  such that  $\lim_{k \rightarrow \infty} (u_{k,0}, u_k) = (u_0, u)$ . Since  $u_{k,0} > 0$  for all  $k \in \mathbb{N}$ , we consider the sequence  $\{u_k/u_{k,0}\}$ . For  $i = 1, \dots, m_1$  and  $j = 1, \dots, m_2$ , we have  $h_i(u_k/u_{k,0}) = h_i^{hom}(u_{k,0}, u_k)/(u_{k,0})^{\deg(h_i)} = 0$  and  $g_j(u_k/u_{k,0}) = g_j^{hom}(u_{k,0}, u_k)/(u_{k,0})^{\deg(g_j)} \geq 0$ . It implies that  $\{u_k/u_{k,0}\} \in S$ . Thus

$$f^{hom}(u_0, u) = \lim_{k \rightarrow \infty} f^{hom}(u_{k,0}, u_k) = \lim_{k \rightarrow \infty} u_{k,0}^{\deg(f)} f(u_k/u_{k,0}) \geq 0,$$

which concludes the proof.  $\square$

Let  $d = \max\{\deg(p), \deg(q)\}$ ,  $\tilde{p}(\tilde{x}) = x_0^d p(x/x_0)$  and  $\tilde{q}(\tilde{x}) = x_0^d q(x/x_0)$ . We reformulate the minimization of (1) as the following constrained polynomial optimization:

$$\begin{cases} s^* := \min_{\tilde{x} \in \mathbb{R}^{n+1}} \tilde{p}(\tilde{x}) \\ \text{s.t. } h_1^{hom}(\tilde{x}) = \dots = h_{m_1}^{hom}(\tilde{x}) = 0, \tilde{q}(\tilde{x}) = 1, \\ g_1^{hom}(\tilde{x}) \geq 0, \dots, g_{m_2}^{hom}(\tilde{x}) \geq 0, x_0 \geq 0. \end{cases}$$

We now investigate the relations between  $r^*$  and  $s^*$ . In the following of this paper, without loss of generality, we always assume that

$$(6) \quad \begin{aligned} &q(x) > 0 \text{ on a neighbourhood of a minimizer of (1); or } q(x) > 0 \text{ for} \\ &\text{any } x \in S \text{ with sufficient large Euclidean norm if } r^* \text{ is not achievable.} \end{aligned}$$

Otherwise we can replace  $\frac{p(x)}{q(x)}$  by  $\frac{p(x)q(x)}{q^2(x)}$ . Note that we do not assume  $q(x)$  is nonnegative on the whole feasible set  $S$  as in [10, 18].

**Definition 2.2.** [15] *If there exists a point  $0 \neq (0, u) \in \tilde{S}$  but  $(0, u) \notin \text{closure}(\tilde{S}_0)$ , then we say  $S$  is not closed at  $\infty$ ; otherwise, we say  $S$  is closed at  $\infty$ .*

**Theorem 2.3.** *It always holds that  $s^* \leq r^*$ , and the equality holds if one of the following conditions is satisfied:*

- (a)  $S$  is closed at  $\infty$ ;
- (b)  $\deg(p) > \deg(q)$ ;
- (c)  $s^*$  is achievable and  $x_0^* > 0$  for at least one of its minimizers  $\tilde{x}^* = (x_0^*, x^*)$ .

*Proof.* We first show that  $s^* \leq r^*$ . For any  $u \in S$  in a neighborhood of a minimizer of (1) or with sufficient large Euclidean norm if  $r^*$  is not achievable, if  $\frac{p(x)}{q(x)}$  is defined at  $u$ , then  $q(u) > 0$  by the assumption in (6). Let  $t = q(u)^{1/d} = \tilde{q}(1, u)^{1/d}$ . We have  $\tilde{q}(1/t, u/t) = 1$  and  $(1/t, u/t) \in \tilde{S}$ , so

$$\frac{p(u)}{q(u)} = \frac{\tilde{p}(1, u)}{\tilde{q}(1, u)} = \frac{\tilde{p}(1/t, u/t)}{\tilde{q}(1/t, u/t)} = \tilde{p}(1/t, u/t) \geq s^*,$$

then we have  $s^* \leq r^*$ . Therefore, to show  $r^* = s^*$ , we only need to show  $r^* \leq s^*$ .

(a) For any feasible point  $(u_0, u)$  of (3), i.e.,  $(u_0, u) \in \tilde{S}$  and  $\tilde{q}(u_0, u) = 1$ , since  $S$  is closed at  $\infty$ , there exists a sequence  $\{(u_{k,0}, u_k)\}$  in  $\tilde{S}$  such that  $u_{k,0} > 0$  for any  $k \in \mathbb{N}$  and  $\lim_{k \rightarrow \infty} (u_{k,0}, u_k) = (u_0, u)$ . Due to the continuity of  $\tilde{q}$ ,  $\lim_{k \rightarrow \infty} \tilde{q}(u_{k,0}, u_k) = 1$ . Hence, we can always assume that for any  $k \in \mathbb{N}$ ,  $\tilde{q}(u_{k,0}, u_k) > 0$ . For each  $k \in \mathbb{N}$ , let  $t_k = \tilde{q}(u_{k,0}, u_k)^{1/d}$  and consider the sequence  $\{(u_{k,0}/t_k, u_k/t_k)\}$ . We have  $\lim_{k \rightarrow \infty} (u_{k,0}/t_k, u_k/t_k) = (u_0, u)$  and  $\tilde{q}(u_{k,0}/t_k, u_k/t_k) = 1$ . For  $i = 1, \dots, m_1$ ,  $j = 1, \dots, m_2$ ,

$$\begin{aligned} 0 &= \frac{1}{t_k^{\deg(h_i)}} h_i^{\text{hom}}(u_{k,0}, u_k) = h_i^{\text{hom}}(u_{k,0}/t_k, u_k/t_k) = \frac{1}{t_k^{\deg(h_i)}} u_{k,0}^{\deg(h_i)} h_i(u_k/u_{k,0}), \\ 0 &\leq \frac{1}{t_k^{\deg(g_j)}} g_j^{\text{hom}}(u_{k,0}, u_k) = g_j^{\text{hom}}(u_{k,0}/t_k, u_k/t_k) = \frac{1}{t_k^{\deg(g_j)}} u_{k,0}^{\deg(g_j)} g_j(u_k/u_{k,0}), \end{aligned}$$

which imply  $(u_{k,0}/t_k, u_k/t_k) \in \tilde{S}$  and  $u_k/u_{k,0} \in S$  for all  $k$ . Hence

$$\tilde{p}(u_{k,0}/t_k, u_k/t_k) = \frac{\tilde{p}(u_{k,0}/t_k, u_k/t_k)}{\tilde{q}(u_{k,0}/t_k, u_k/t_k)} = \frac{p(u_k/u_{k,0})}{q(u_k/u_{k,0})} \geq r^*$$

and  $\tilde{p}(u_0, u) = \lim_{k \rightarrow \infty} \tilde{p}(u_{k,0}/t_k, u_k/t_k) \geq r^*$  which means  $r^* \leq s^*$ .

(b) If  $\deg(p) > \deg(q)$ , then  $x_0$  divides  $\tilde{q}(\tilde{x})$ . By  $\tilde{q}(\tilde{x}) = 1$ , we have  $u_0 > 0$  for any feasible point  $(u_0, u)$  of (3) and it is easy to see that  $u/u_0 \in S$ , then

$$\tilde{p}(u_0, u) = \frac{\tilde{p}(u_0, u)}{\tilde{q}(u_0, u)} = \frac{\tilde{p}(1, u/u_0)}{\tilde{q}(1, u/u_0)} = \frac{p(u/u_0)}{q(u/u_0)} \geq r^*,$$

which means  $r^* \leq s^*$ .

(c) Since  $x_0^* > 0$ , we have  $x^*/x_0^* \in S$  and

$$s^* = \tilde{p}(x_0^*, x^*) = \frac{\tilde{p}(x_0^*, x^*)}{\tilde{q}(x_0^*, x^*)} = \frac{p(x^*/x_0^*)}{q(x^*/x_0^*)} \geq r^*,$$

which implies  $r^* = s^*$ .  $\square$

The following corollary shows that the minimizations of (1) and (3) are always equivalent when there are no constraints in (1).

**Corollary 2.4.** *If  $m_1 = m_2 = 0$  in (1), then  $S = \mathbb{R}^n$  is closed at  $\infty$  and  $r^* = s^*$ .*

**Remark 2.5.** If  $S = \mathbb{R}^n$ , we can remove  $x_0 \geq 0$  in (3). In fact, if there are no constraints, according to the proof of Part (a) in Theorem 2.3, we only need  $u_{k,0} \neq 0$  to get the same result. Therefore, the global minimization

$$r^* := \min_{x \in \mathbb{R}^n} \frac{p(x)}{q(x)}$$

is equivalent to

$$(7) \quad \begin{cases} s^* := \min_{\tilde{x} \in \mathbb{R}^{n+1}} \tilde{p}(\tilde{x}) \\ \text{s.t. } \tilde{q}(\tilde{x}) = 1. \end{cases}$$

**Remark 2.6.** We would like to point out that not every  $S$  is closed at  $\infty$  and  $s^*$  might be strictly smaller than  $r^*$  in this case. For example, consider the following problem:

$$(8) \quad \begin{cases} r^* := \min_{x_1, x_2 \in \mathbb{R}} \frac{x_1}{(x_1 - x_2)^2} \\ \text{s.t. } x_1^2(x_1 - x_2) = 1, \\ x_1 - 1 \geq 0. \end{cases}$$

Clearly, we have  $r^* = 1$ . However, [17, Example 5.2 (i)] shows that the set

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2(x_1 - x_2) - 1 = 0\}$$

is not closed at  $\infty$ . Actually,

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2(x_1 - x_2) - 1 = 0, x_1 - 1 \geq 0\}$$

is not closed at  $\infty$ , either. To see it, we have

$$\tilde{S} := \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^2(x_1 - x_2) - x_0^3 = 0, x_1 - x_0 \geq 0, x_0 \geq 0\}.$$

Consider the point  $(0, 0, 1) \in \tilde{S}$ . Suppose that there exists a sequence  $\{(x_{k,0}, x_{k,1}, x_{k,2})\}$  in  $\tilde{S}$  such that  $\lim_{k \rightarrow \infty} (x_{k,0}, x_{k,1}, x_{k,2}) = (0, 0, 1)$  and  $x_{k,0} > 0$  for all  $k \in \mathbb{N}$ . Then for  $0 < \varepsilon < 1/2$ , there exists  $N \in \mathbb{N}$  such that for any  $k > N$ , we have

$$0 < x_{k,0} < \varepsilon, \quad |x_{k,1}| < \varepsilon, \quad |x_{k,2} - 1| < \varepsilon.$$

Thus

$$0 < x_{k,0}^3 = x_{k,1}^2(x_{k,1} - x_{k,2}) \leq x_{k,1}^2(\varepsilon - 1 + \varepsilon) \leq 0$$

which is a contradiction. Therefore,  $S$  is not closed at  $\infty$  and we have  $s^* = 0 < r^*$  if we reformulate (8) by homogenization as the following problem

$$\begin{cases} s^* := \min_{x_0, x_1, x_2 \in \mathbb{R}} x_0 x_1 \\ \text{s.t. } (x_1 - x_2)^2 - 1 = x_1^2(x_1 - x_2) - x_0^3 = 0, \\ x_1 - x_0 \geq 0, x_0 \geq 0. \end{cases}$$

However, in section 2.2 we will show that the closedness at  $\infty$  is a generic condition for a given set  $S$ .

Let

$$\hat{S} := \{x \in \mathbb{R}^n \mid \hat{h}_i(x) = 0, \hat{g}_j(x) \geq 0, i = 1, \dots, m_1, j = 1, \dots, m_2\}$$

where  $\hat{h}_i$  and  $\hat{g}_j$  denote the homogeneous parts of the highest degree of  $h_i$  and  $g_j$ , respectively. Denote  $p_d(x)$  and  $q_d(x)$  the homogeneous parts of degree  $d$  of  $p(x)$  and  $q(x)$ , respectively.

**Theorem 2.7.** *If one of the conditions in Theorem 2.3 holds, then the following properties hold.*

- (a)  $r^*$  is achievable if and only if  $s^*$  is achievable at a minimizer  $\tilde{x}^* = (x_0^*, x^*)$  with  $x_0^* \neq 0$ ;
- (b) If neither  $p(x)$  and  $q(x)$  have real common roots in  $S$ , nor  $p_d(x)$  and  $q_d(x)$  have real nonzero common roots in  $\hat{S}$ , then  $s^*$  is achievable.
- (c) If  $s^*$  is achievable and  $x_0^* = 0$  for all minimizers  $\tilde{x}^* = (x_0^*, x^*)$  of (3), then  $r^*$  is not achievable. For each minimizer  $\tilde{x}^* = (0, x^*)$  of (3), if there exists a sequence  $\{\tilde{x}_k\} = \{(x_{k,0}, x_k)\}$  in  $\tilde{S}$  such that  $\lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}^*$  and  $x_{k,0} > 0$  for all  $k \in \mathbb{N}$ , then  $\lim_{k \rightarrow \infty} \frac{p(x_k/x_{k,0})}{q(x_k/x_{k,0})} = r^*$ .

*Proof.* If one of the conditions in Theorem 2.3 holds, we have  $r^* = s^*$ .

(a) Let  $x^*$  be a minimizer of (1), then  $x^* \in S$  and  $t = \tilde{q}(1, x^*)^{1/d} = q(x^*)^{1/d} > 0$  by the assumption in (6). It is easy to verify that  $(1/t, x^*/t) \in \tilde{S}$  and  $\tilde{q}(1/t, x^*/t) = 1$ . We have  $\tilde{p}(1/t, x^*/t) = r^* = s^*$  which means  $(1/t, x^*/t)$  is a minimizer of (3). If  $s^*$  is achieved at  $\tilde{x}^* = (x_0^*, x^*) \in \tilde{S}$  with  $x_0^* > 0$ , then  $r^*$  is achieved at  $x^*/x_0^* \in S$ .

(b) To the contrary, we assume that  $s^*$  is not achievable. Then there exists a sequence  $\{\tilde{x}_k\}$  in  $\tilde{S}$  such that  $\lim_{k \rightarrow \infty} \|\tilde{x}_k\|_2 = \infty$ ,  $\lim_{k \rightarrow \infty} \tilde{p}(\tilde{x}_k) = s^*$  and for all  $k \in \mathbb{N}$ ,  $\tilde{q}(\tilde{x}_k) = 1$ . Consider the bounded sequence  $\{\tilde{x}_k/\|\tilde{x}_k\|_2\} \subseteq \tilde{S}$ . By Bolzano-Weierstrass Theorem, there exists a subsequence  $\{\tilde{x}_{k_j}/\|\tilde{x}_{k_j}\|_2\}$  such that  $\lim_{j \rightarrow \infty} \tilde{x}_{k_j}/\|\tilde{x}_{k_j}\|_2 = \tilde{y}$  for some nonzero  $\tilde{y} = (y_0, y) \in \tilde{S}$  since  $\tilde{S}$  is closed. Let  $\tilde{p}(\tilde{x}_{k_j}) = s_{k_j}$ , then  $\lim_{j \rightarrow \infty} s_{k_j} = s^*$ . Since  $\tilde{p}(\tilde{x}_{k_j}) = (\|\tilde{x}_{k_j}\|_2)^d \tilde{p}(\tilde{x}_{k_j}/\|\tilde{x}_{k_j}\|_2)$  and  $\lim_{j \rightarrow \infty} \|\tilde{x}_{k_j}\|_2 = \infty$ , we have  $\tilde{p}(\tilde{y}) = \lim_{j \rightarrow \infty} \tilde{p}(\tilde{x}_{k_j}/\|\tilde{x}_{k_j}\|_2) = 0$ . Similarly, we can prove  $\tilde{q}(\tilde{y}) = \lim_{j \rightarrow \infty} \tilde{q}(\tilde{x}_{k_j}/\|\tilde{x}_{k_j}\|_2) = 0$ . Thus  $\tilde{p}(\tilde{y})$  and  $\tilde{q}(\tilde{y})$  have real nonzero common root  $\tilde{y}$  on unit sphere  $S^{n+1}$ . We have  $y_0 = 0$ , otherwise  $y/y_0$  is a real common root of  $p(x)$  and  $q(x)$  in  $S$ . Therefore  $0 = \tilde{p}(\tilde{y}) = p_d(y)$ ,  $0 = \tilde{q}(\tilde{y}) = q_d(y)$ ,  $0 = h_i^{hom}(\tilde{y}) = \hat{h}_i(y)$  and  $0 \leq g_j^{hom}(\tilde{y}) = \hat{g}_j(y)$ , i.e.  $p_d(x)$  and  $q_d(x)$  have real nonzero common root  $y$  in  $\hat{S}$  which is a contradiction.

(c) By (a), if  $x_0^* = 0$  for all minimizers of (3),  $r^*$  is not achievable. Suppose  $\tilde{x}^* = (0, x^*)$  is a minimizer of (3) and there exists a sequence  $\{\tilde{x}_k\} = \{(x_{k,0}, x_k)\}$  in  $\tilde{S}$  such that  $\lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}^*$  and  $x_{k,0} > 0$ . Then for each  $k \in \mathbb{N}$ ,  $x_k/x_{k,0} \in S$ . Since  $\tilde{p}$  and  $\tilde{q}$  are continuous,  $\lim_{k \rightarrow \infty} \tilde{p}(x_{k,0}, x_k) = s^*$  and  $\lim_{k \rightarrow \infty} \tilde{q}(x_{k,0}, x_k) = 1$ . Therefore,

$$\lim_{k \rightarrow \infty} \frac{p(x_k/x_{k,0})}{q(x_k/x_{k,0})} = \lim_{k \rightarrow \infty} \frac{\tilde{p}(x_{k,0}, x_k)}{\tilde{q}(x_{k,0}, x_k)} = s^* = r^*.$$

Here completes the proof.  $\square$

**2.2. On the generality of closedness at infinity.** Although we have counter example in Remark 2.6, we next show that in general a given set  $S$  in (5) is indeed closed at  $\infty$ . Therefore, if the constraints in (1) are generic, (1) and (3) are equivalent.

Let us first review some elementary background about *resultants* and *discriminants*. More details can be found in [2, 4, 17]. Let  $f_1, \dots, f_n$  be homogeneous polynomials in  $x = (x_1, \dots, x_n)$ . The resultant  $Res(f_1, \dots, f_n)$  is a polynomial in

the coefficients of  $f_1, \dots, f_n$  satisfying

$$\text{Res}(f_1, \dots, f_n) = 0 \Leftrightarrow \exists 0 \neq u \in \mathbb{C}^n, f_1(u) = \dots = f_n(u) = 0.$$

Let  $f_1, \dots, f_m$  be homogeneous polynomials with  $m < n$ . The discriminant for  $f_1, \dots, f_m$ , denoted by  $\Delta(f_1, \dots, f_m)$ , is a polynomial in the coefficients of  $f_1, \dots, f_m$  such that

$$\Delta(f_1, \dots, f_m) = 0$$

if and only if the polynomial system

$$f_1(x) = \dots = f_m(x) = 0$$

has a solution  $0 \neq u \in \mathbb{C}^n$  such that the Jacobian matrix of  $f_1, \dots, f_n$  does not have full rank.

We next show that in general a given set  $S$  in (5) is closed at  $\infty$ . In the following, we suppose  $S$  is not closed at  $\infty$  and fix a nonzero point  $(0, u) \in \tilde{S}$  but  $(0, u) \notin \text{closure}(\tilde{S}_0)$ . Let  $J(u) := \{j \in [m_2] \mid g_j^{\text{hom}}(0, u) = 0\}$ . Then  $g_j^{\text{hom}}(0, u) > 0$  for all  $j \in [m_2] \setminus J(u)$ . We have the cardinality  $m_1 + |J(u)| \geq 1$ , otherwise  $(0, u)$  is an interior point of  $\tilde{S}$  and  $(0, u) \in \text{closure}(\tilde{S}_0)$ . Let

$$V(u) := \{\tilde{x} \in \mathbb{R}^{n+1} \mid h_i^{\text{hom}}(\tilde{x}) = 0, g_j^{\text{hom}}(\tilde{x}) = 0, i \in [m_1], j \in J(u)\}.$$

For any  $\delta > 0$ , let

$$B((0, u), \delta) = \{(x_0, x) \in \mathbb{R}^{n+1} \mid \|(x_0, x) - (0, u)\|_2 \leq \delta\}.$$

**Lemma 2.8.** *Suppose  $S$  is not closed at  $\infty$ , then there exists  $\delta > 0$  such that for all  $(x_0, x) \in B((0, u), \delta) \cap V(u)$ , we have  $x_0 \leq 0$ .*

*Proof.* Suppose such  $\delta$  doesn't exist. Consider a sequence  $\{\delta_k\}$  with  $\delta_k > 0$  and  $\lim_{k \rightarrow \infty} \delta_k = 0$ . Then for each  $k$ , there exists a point  $(u_{k,0}, u_k) \in B((0, u), \delta_k) \cap V(u)$  such that  $u_{k,0} > 0$ . By the continuity, there exists  $N$  such that for all  $k \geq N$ ,  $g_j^{\text{hom}}(u_{k,0}, u_k) > 0$  for each  $j \in [m_2] \setminus J(u)$  which implies  $(u_{k,0}, u_k) \in \tilde{S}$  for all  $k \geq N$  and  $(0, u) \in \text{closure}(\tilde{S}_0)$ . The contradiction follows.  $\square$

Now let us recall the Implicit Function Theorem.

**Theorem 2.9.** [12, Theorem 3.3.1; THE IMPLICIT FUNCTION THEOREM] *Let*

$$\Phi(x) = \Phi(x_1, \dots, x_n) \equiv (\phi_1(x_1, \dots, x_n), \dots, \phi_m(x_1, \dots, x_n))$$

*be a mapping of class  $C^k$ ,  $k \geq 1$ , defined on an open set  $U \subseteq \mathbb{R}^n$  and taking values in  $\mathbb{R}^m$ . Assume that  $1 \leq m < n$ . Let  $x^0 = (x_1^0, \dots, x_n^0)$  be a fixed point of  $U$  and  $x_a^0 = (x_1^0, \dots, x_{n-m}^0)$ . Suppose that the Jacobian determinant*

$$\frac{\partial(\phi_1, \dots, \phi_m)}{\partial(x_{n-m+1}, \dots, x_n)}(x^0) \neq 0.$$

*Then there exists a neighborhood  $\tilde{U}$  of  $x^0$ , and open set  $W \subseteq \mathbb{R}^{n-m}$  containing  $x_a^0$ , and functions  $f_1, \dots, f_m$  of class  $C^k$  on  $W$  such that*

$$\Phi(x_1, \dots, x_{n-m}, f_1(x_a), \dots, f_m(x_a)) = 0 \quad \text{for every } x_a \in W.$$

*Here,  $x_a = (x_1, \dots, x_{n-m})$ . Furthermore,  $f_1, \dots, f_m$  are the unique functions satisfying*

$$\{x \in \tilde{U} \mid \Phi(x) = 0\} = \{x \in \tilde{U} \mid x_a \in W, x_{n-m+k} = f_k(x_a) \text{ for } k = 1, \dots, m\}.$$



Let  $J(u) = \{j_1, \dots, j_l\}$  and

$$A(u) := \begin{bmatrix} \frac{\partial h_1^{hom}}{\partial x_1}(0, u) & \cdots & \frac{\partial h_1^{hom}}{\partial x_n}(0, u) \\ \vdots & & \vdots \\ \frac{\partial h_{m_1}^{hom}}{\partial x_1}(0, u) & \cdots & \frac{\partial h_{m_1}^{hom}}{\partial x_n}(0, u) \\ \frac{\partial g_{j_1}^{hom}}{\partial x_1}(0, u) & \cdots & \frac{\partial g_{j_1}^{hom}}{\partial x_n}(0, u) \\ \vdots & & \vdots \\ \frac{\partial g_{j_l}^{hom}}{\partial x_1}(0, u) & \cdots & \frac{\partial g_{j_l}^{hom}}{\partial x_n}(0, u) \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{h}_1}{\partial x_1}(u) & \cdots & \frac{\partial \hat{h}_1}{\partial x_n}(u) \\ \vdots & & \vdots \\ \frac{\partial \hat{h}_{m_1}}{\partial x_1}(u) & \cdots & \frac{\partial \hat{h}_{m_1}}{\partial x_n}(u) \\ \frac{\partial \hat{g}_{j_1}}{\partial x_1}(u) & \cdots & \frac{\partial \hat{g}_{j_1}}{\partial x_n}(u) \\ \vdots & & \vdots \\ \frac{\partial \hat{g}_{j_l}}{\partial x_1}(u) & \cdots & \frac{\partial \hat{g}_{j_l}}{\partial x_n}(u) \end{bmatrix}$$

Recall that  $\hat{h}_i$  and  $\hat{g}_j$  denote the homogeneous parts of the highest degree of  $h_i$  and  $g_j$ , respectively. Combining Lemma 2.8 and the Implicit Function Theorem, we have

**Lemma 2.10.** *Suppose  $S$  is not closed at  $\infty$  and  $m_1 + |J(u)| < n + 1$ , then  $\text{rank } A(u) < m_1 + |J(u)|$ .*

*Proof.* Let  $m = m_1 + |J(u)|$ . Suppose  $\text{rank } A(u) = m$ . Then there exist  $m$  independent columns in  $A(u)$ . Without loss of generality, we assume the last  $m$  columns of  $A(u)$  are independent, i.e. the Jacobian determinant

$$\frac{\partial(h_1^{hom}, \dots, h_{m_1}^{hom}, g_{j_1}^{hom}, \dots, g_{j_l}^{hom})}{\partial(x_{n-m+1}, \dots, x_n)}(0, u) \neq 0.$$

Partition  $\tilde{u} = (0, u)$  as  $(\tilde{u}^a, \tilde{u}^b)$  where  $\tilde{u}^a = (0, u_1, \dots, u_{n-m})$ ,  $\tilde{u}^b = (u_{n-m+1}, \dots, u_n)$ . Then by the Implicit Function Theorem 2.9, there exists an open set  $W \subseteq \mathbb{R}^{n-m+1}$  containing  $\tilde{u}^a$ , and functions  $f_1, \dots, f_m$  of class  $\mathcal{C}^k$  on  $W$  such that

$$\begin{aligned} h_i^{hom}(x_0, \dots, x_{n-m}, f_1(\tilde{x}^a), \dots, f_m(\tilde{x}^a)) &= 0, \quad i = 1, \dots, m_1, \\ g_j^{hom}(x_0, \dots, x_{n-m}, f_1(\tilde{x}^a), \dots, f_m(\tilde{x}^a)) &= 0, \quad j \in J(u), \end{aligned}$$

for every  $\tilde{x}^a \in W$ . Here,  $\tilde{x}^a = (x_0, \dots, x_{n-m})$ . Therefore,  $(\tilde{x}^a, f_1(\tilde{x}^a), \dots, f_m(\tilde{x}^a)) \in V(u)$  for every  $\tilde{x}^a \in W$ . Since  $W$  is open and  $f_1, \dots, f_m$  are continuous, we can choose  $\tilde{x}^a$  very close to  $\tilde{u}^a$  such that  $(\tilde{x}^a, f_1(\tilde{x}^a), \dots, f_m(\tilde{x}^a)) \in B((0, u), \delta) \cap V(u)$  with  $x_0 > 0$  for any  $\delta > 0$ , which contradicts the conclusion in Lemma 2.8.  $\square$

The following theorem shows that if the defining polynomials of  $S$  are generic, then  $S$  is closed at  $\infty$ .

**Theorem 2.11.** *Suppose  $S$  is not closed at  $\infty$ , then*

- (a) *if  $m_1 + |J(u)| \geq n + 1$ , then  $\text{Res}(h_1^{hom}, \dots, h_{m_1}^{hom}, g_{j_1}^{hom}, \dots, g_{j_{n-m_1+1}}^{hom}) = 0$  for every  $\{j_1, \dots, j_{n-m_1+1}\} \subseteq J(u)$ ;*
- (b) *if  $m_1 + |J(u)| = n$ , then  $\text{Res}(\hat{h}_1, \dots, \hat{h}_{m_1}, \hat{g}_{j_1}, \dots, \hat{g}_{j_l}) = 0$ ;*
- (c) *if  $m_1 + |J(u)| < n$ , then  $\Delta(\hat{h}_1, \dots, \hat{h}_{m_1}, \hat{g}_{j_1}, \dots, \hat{g}_{j_l}) = 0$ .*

*Proof.* Since  $h_i^{hom}(0, u) = \hat{h}_i(u) = 0$ ,  $g_j^{hom}(0, u) = \hat{g}_j(u) = 0$  for all  $i \in [m_1], j \in J(u)$ , then the conclusions in (a) and (b) are implied by the proposition of resultants. If  $m_1 + |J(u)| < n$ , then by Lemma 2.10, the Jacobian matrix of  $(\hat{h}_1, \dots, \hat{h}_{m_1}, \hat{g}_{j_1}, \dots, \hat{g}_{j_l})$  does not have full rank at  $u$ . Hence, the conclusion in (c) follows by the proposition of discriminants.  $\square$

In this section, we reformulate the minimization of (1) as the polynomial optimization (3) by homogenization. Suppose  $S$  is closed at  $\infty$  which is generic and always true when  $S = \mathbb{R}^n$ , then  $r^* = s^*$ . The relations between the achievabilities of  $r^*$  and  $s^*$  are discussed in Proposition 2.7. Now the problem becomes how to efficiently solve polynomial optimization (3). Recently, there has been much work on solving polynomial optimization with or without constraints via SOS relaxation. In next section, we introduce the Jacobian SDP relaxation [15] and show that the assumptions under which the Jacobian SDP relaxation is exact can be weakened.

### 3. JACOBIAN SDP RELAXATION APPLICABLE TO FINITE REAL SINGULARITIES

Consider the following polynomial optimization problem

$$(9) \quad \begin{cases} f_{\min} := \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } h_1(x) = \cdots = h_{m_1}(x) = 0, \\ g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0. \end{cases}$$

where  $f(x), h_i(x), g_j(x) \in \mathbb{R}[x_1, \dots, x_n]$ . In this section, we first introduce the exact Jacobian SDP relaxation proposed in [15]. Then we present our contribution in this section by giving a weakened assumption under which the relaxation in [15] is still exact.

Let  $m = \min\{m_1 + m_2, n - 1\}$ . For convenience, denote  $h(x) = (h_1(x), \dots, h_{m_1}(x))$  and  $g(x) = (g_1(x), \dots, g_{m_2}(x))$ . For a subset  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$ , denote  $g_J(x) = (g_{j_1}(x), \dots, g_{j_k}(x))$ . Symbols  $\nabla h(x)$  and  $\nabla g_J(x)$  represent the gradient vectors of the polynomials in  $h(x)$  and  $g_J(x)$ , respectively. Denote the determinantal variety of  $(f, h, g_J)$ 's Jacobian being singular by

$$G_J = \{x \in \mathbb{C}^n \mid \text{rank } B^J(x) \leq m_1 + |J|\}, \quad B^J(x) = [\nabla f(x) \quad \nabla h(x) \quad \nabla g_J(x)].$$

For every  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$  with  $k \leq m - m_1$ , let  $\eta_1^J, \dots, \eta_{\text{len}(J)}^J$  be the set of defining polynomials for  $G_J$  where  $\text{len}(J)$  is the number of these polynomials. See [15, Section 2.1] about minimizing the number of defining equations for determinantal varieties. For each  $i = 1, \dots, \text{len}(J)$ , define

$$(10) \quad \varphi_i^J(x) = \eta_i^J \cdot \prod_{j \in J^c} g_j(x), \text{ where } J^c = [m_2] \setminus J.$$

For simplicity, we list all possible  $\varphi_i^J$  in (10) sequentially as

$$(11) \quad \varphi_1, \varphi_2, \dots, \varphi_r, \text{ where } r = \sum_{J \subseteq [m_2], |J| \leq m - m_1} \text{len}(J).$$

Define the variety

$$(12) \quad W := \{x \in \mathbb{C}^n \mid h_1(x) = \cdots = h_{m_1}(x) = \varphi_1(x) = \cdots = \varphi_r(x) = 0\}.$$

We consider the following optimization

$$(13) \quad \begin{cases} f^* := \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } h_i(x) = 0 \ (i = 1, \dots, m_1), \ \varphi_j(x) = 0 \ (j = 1, \dots, r), \\ g_\nu(x) \geq 0, \forall \nu \in \{0, 1\}^{m_2}, \end{cases}$$

where  $g_\nu = g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$ .

We now construct  $N$ -th order SDP relaxation [13] for (13) and its dual problem. Let  $\psi(x)$  be a polynomial with  $\deg(\psi) \leq 2N$  and define symmetric matrices  $A_\alpha^{(N)}$  such that

$$\psi(x)[x]_d[x]_d^T = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq 2N} A_\alpha^{(N)} x^\alpha, \text{ where } d = N - \lceil \deg(\psi)/2 \rceil.$$

Then the  $N$ -th order localizing moment matrix of  $\psi$  is defined as

$$(14) \quad L_\psi^{(N)}(y) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq 2N} A_\alpha^{(N)} y_\alpha,$$

where  $y$  is a moment vector indexed by  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \leq 2N$ . Denote

$$L_f(y) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq \deg(f)} f_\alpha y_\alpha \quad \text{for} \quad f(x) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq \deg(f)} f_\alpha x^\alpha.$$

The  $N$ -th order SDP relaxation [13] for (13) is the SDP

$$(15) \quad \begin{cases} f_N^{(1)} := \min L_f(y) \\ \text{s.t. } L_{h_i}^{(N)}(y) = 0 \ (i = 1, \dots, m_1), \ L_{\varphi_j}^{(N)}(y) = 0 \ (j = 1, \dots, r), \\ L_{g_\nu}^{(N)} \succeq 0, \forall \nu \in \{0, 1\}^{m_2}, y_0 = 1. \end{cases}$$

Now we present the dual of (15). Define the truncated preordering  $P^{(N)}$  generated by  $g_j$  as

$$P^{(N)} = \left\{ \sum_{\nu \in \{0, 1\}^{m_2}} \sigma_\nu g_\nu \mid \deg(\sigma_\nu g_\nu) \leq 2N, \sigma_\nu \text{'s are SOS} \right\},$$

and the truncated ideal  $I^{(N)}$  generated by  $h_i$  and  $\varphi_j$  as

$$I^{(N)} = \left\{ \sum_{i=1}^{m_1} \psi_i h_i + \sum_{j=1}^r \phi_j \varphi_j \mid \begin{array}{l} \deg(\psi_i h_i) \leq 2N \ \forall i \\ \deg(\phi_j \varphi_j) \leq 2N \ \forall j \end{array} \right\}.$$

It is shown [13] that the dual of (15) is the following SOS relaxation for (13):

$$(16) \quad \begin{cases} f_N^{(2)} := \max \gamma \\ \text{s.t. } f(x) - \gamma \in I^{(N)} + P^{(N)}. \end{cases}$$

By weak duality, we have  $f_N^{(2)} \leq f_N^{(1)} \leq f^*$ . For any subset  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$ , let

$$V(h, g_J) = \{x \in \mathbb{C}^n \mid h_i(x) = 0, \ g_j(x) = 0, \ i = 1, \dots, m_1, \ j \in J\}.$$

We make the following assumption.

**Assumption 3.1.** (i)  $m_1 \leq n$ . (ii) For any feasible point  $u$ , at most  $n - m_1$  of  $g_1(u), \dots, g_{m_2}(u)$  vanish. (iii) For every  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$  with  $k \leq n - m_1$ , Jacobian  $[\nabla h \ \nabla g_J]$  has full rank on  $V(h, g_J)$ .

Under the above assumption, the following main result is shown in [15].

**Theorem 3.2.** [15, Theorem 2.3] Suppose Assumption 3.1 holds. Then  $f^* > -\infty$  and there exists  $N^* \in \mathbb{N}$  such that  $f_N^{(1)} = f_N^{(2)} = f^*$  for all  $N \geq N^*$ . Furthermore, if the minimum  $f_{\min}$  of (9) is achievable, then  $f_N^{(1)} = f_N^{(2)} = f_{\min}$  for all  $N \geq N^*$ .

According to Theorem 3.2, it is possible to solve the polynomial optimization (9) *exactly* by a single SDP relaxation, which was not known in the prior existing literature. It is also shown in [15] that Assumption 3.1 is generically true. It is the reason why we use this method to solve (3). We will show later in this paper that Assumption 3.1 is always true for (3) when the original feasible set  $S = \mathbb{R}^n$ , i.e. for the global minimization of a rational function. In the following of this section, we prove that the condition (iii) in Assumption 3.1 can always be weakened such that the conclusions in Theorem 3.2 still hold.

**Definition 3.3.** For every set  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$  with  $k \leq n - m_1$ , let

$$\Theta_J = \{x \in V(h, g_J) \mid \text{rank} [\nabla h \quad \nabla g_J] < m_1 + |J|\} \quad \text{and} \quad \Theta = \bigcup_{J \subseteq [m_2], |J| \leq n - m_1} \Theta_J.$$

We next show that the Jacobian SDP relaxation [15] is still exact under the following weakened assumption:

**Assumption 3.4.** (i)  $m_1 \leq n$ . (ii) For any  $u \in S$ , at most  $n - m_1$  of  $g_1(u), \dots, g_{m_2}(u)$  vanish. (iii) The set  $\Theta$  is finite.

Let  $K$  be the variety defined by the KKT conditions

$$K = \left\{ (x, \lambda, \mu) \in \mathbb{C}^{n+m_1+m_2} \left| \begin{array}{l} \nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x) \\ h_i(x) = \mu_j g_j(x) = 0, \forall (i, j) \in [m_1] \times [m_2] \end{array} \right. \right\}$$

and

$$K_x = \{x \in \mathbb{C}^n \mid (x, \lambda, \mu) \in K \text{ for some } \lambda, \mu\}.$$

Under Assumption 3.1, [15, Lemma 3.1] states that  $W = K_x$ . We now improve this result as follows.

**Lemma 3.5** (Revised Version of Lemma 3.1 in [15]). *Under conditions (i) and (ii) in Assumption 3.4,  $W = \Theta \cup K_x$ .*

*Proof.* The proof of [15, Lemma 3.1] shows that  $W \setminus \Theta \subseteq K_x \subseteq W$ . With a similar argument, we prove  $\Theta \subseteq W$ . Recall that  $B^J = [\nabla f(x) \quad \nabla h(x) \quad \nabla g_J(x)]$ . Choose an arbitrary  $u \in \Theta$  and let  $u \in \Theta_I$  for some  $I \subseteq [m_2]$ . If  $I = \emptyset$ , then  $[\nabla h]$  and  $B^J(u)$  are both singular for any  $J \subseteq [m_2]$ , which implies  $\varphi_i(u) = 0$  and  $u \in W$ . If  $I \neq \emptyset$ , write  $I = \{i_1, \dots, i_t\}$ . Let  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$  be an arbitrary index set with  $m_1 + k \leq m$ .

**Case  $I \not\subseteq J$**  At least one  $j \in J^c$  belongs to  $I$ . By the choice of  $I$  and the definition of  $\varphi_i(x)$ ,

$$\varphi_i^J(u) = \eta_i^J \cdot \prod_{j \in J^c} g_j(u) = 0.$$

**Case  $I \subseteq J$**  Then  $[\nabla h \quad \nabla g_I]$  and  $[\nabla f(x) \quad \nabla h(x) \quad \nabla g_J(x)]$  are both singular. Hence, all polynomials  $\varphi_i^J(x)$ 's vanish at  $u$ .

Combining the above two cases, we have all  $\varphi_i^J(x)$  vanish at  $u$ . Thus,  $u \in W$  which implies  $W = \Theta \cup K_x$ .  $\square$

**Lemma 3.6.** *Under conditions (i) and (ii) in Assumption 3.4, if the minimum  $f_{\min}$  of (9) is achievable, then  $f^* = f_{\min}$ .*

*Proof.* By the construction of (13),  $f^* \geq f_{\min}$ . Suppose  $f_{\min} = f(x^*)$  where  $x^*$  is a feasible point of (9). If  $x^* \notin \Theta$ , then the linear independence constraint qualification (LICQ) is satisfied at  $x^*$  which implies  $x^* \in K_x$  [20, Theorem 12.1]. Since  $W = \Theta \cup K_x$  by Lemma 3.5, we have  $x^* \in W$  which implies  $f^* = f_{\min}$ .  $\square$

Next we show that the conclusion in [15, Lemma 3.2] still holds under Assumption 3.4.

**Lemma 3.7** (Revised Version of Lemma 3.2 in [15]). *Suppose Assumption 3.4 holds. Let  $T = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0, j = 1, \dots, m_2\}$ . Then there exist disjoint subvarieties  $W_0, W_1, \dots, W_r$  of  $W$  and distinct  $v_1, \dots, v_r \in \mathbb{R}$  such that*

$$W = W_0 \cup W_1 \cup \dots \cup W_r, \quad W_0 \cap T = \emptyset, \quad W_i \cap T \neq \emptyset, \quad i = 1, \dots, r,$$

*and  $f(x)$  is constantly equal to  $v_i$  on  $W_i$  for  $i = 1, \dots, r$ .*

*Proof.* Denote  $\text{Zar}(K_x)$  the Zariski closure of  $K_x$  and let  $\Omega = W \setminus \text{Zar}(K_x)$ . By Lemma 3.5, we have  $\text{Zar}(K_x) \subseteq W$  and  $\Omega \subseteq \Theta$ . With the proof of [15, Lemma 3.2], we can conclude that there exist disjoint subvarieties  $W_0, W_1, \dots, W_t$  of  $\text{Zar}(K_x)$  and distinct  $v_1, \dots, v_t \in \mathbb{R}$  such that

$$\text{Zar}(K_x) = W_0 \cup W_1 \cup \dots \cup W_t, \quad W_0 \cap T = \emptyset, \quad W_i \cap T \neq \emptyset, \quad i = 1, \dots, t,$$

and  $f(x)$  is constantly equal to  $v_i$  on  $W_i$  for  $i = 1, \dots, t$ . We now consider the set  $\Omega$ . Let  $W_0 = V(E_0)$ , then for any  $u \in \Omega \cap \mathbb{C}^n$ ,  $W_0 \cup \{u\} = V(E_0) \cup V(\langle x - u \rangle) = V(\langle x - u \rangle \cdot E_0)$ . Since  $\Omega \cap \mathbb{C}^n \subseteq \Theta$  is a finite set by Assumption 3.4, if we group  $W_0$  and  $\Omega \cap \mathbb{C}^n$  together then we get a new subvariety. We still denote it by  $W_0$  for convenience. Then  $W_0 \cap T = \emptyset$ . Take any  $w \in \Omega \cap \mathbb{R}^n$ , if  $f(w) = v_{i_0}$  for some  $i_0 \in \{1, \dots, t\}$ , then we put  $w$  into  $W_{i_0}$  and get a new subvariety by the same reason as  $W_0$ . We still write the resulting subvariety as  $W_{i_0}$ . If for any  $i \in \{1, \dots, t\}$ ,  $f(w) \neq v_i$ , then let  $W_{t+1} = \{w\}$  and  $v_{t+1} = f(w) \in \mathbb{R}$ . Since  $\Omega \cap \mathbb{R}^n \subseteq \Theta$  is a finite set, the above process will terminate and we can obtain the required decomposition of  $W$ .  $\square$

Since we get the same result as in [15, Lemma 3.2] under the weakened Assumption 3.4, [15, Theorem 3.4] which is based on [15, Lemma 3.2] can be restated as follows.

**Theorem 3.8** (Revised Version of Theorem 3.4 in [15]). *Suppose Assumption 3.4 holds. Then  $f^* > -\infty$  and there exists  $N^* \in \mathbb{N}$  such that for all  $\varepsilon > 0$*

$$(17) \quad f(x) - f^* + \varepsilon \in I^{(N^*)} + P^{(N^*)}.$$

Since  $\varepsilon$  in (17) is arbitrary, by Lemma 3.6, Theorem 3.2 becomes

**Theorem 3.9** (Revised Version of Theorem 2.3 in [15]). *Suppose Assumption 3.4 holds. Then  $f^* > -\infty$  and there exists  $N^* \in \mathbb{N}$  such that  $f_N^{(1)} = f_N^{(2)} = f^*$  for all  $N \geq N^*$ . Furthermore, if the minimum  $f_{\min}$  of (9) is achievable, then  $f_N^{(1)} = f_N^{(2)} = f_{\min}$  for all  $N \geq N^*$ .*

**Remark 3.10.** *We now compare the conditions (iii) in Assumption 3.1 and 3.4. For any  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$  with  $k \leq n - m_1$ , suppose the ideal  $\langle h, g_J \rangle$  is radical and its codimension is  $m_1 + |J|$ . Then the condition (iii) in Assumption 3.1 requires the variety  $V(h, g_J)$  is nonsingular for every subset  $J$ . In this section, we have proved that if the singularities of  $V(h, g_J)$  are finite, i.e. the condition (iii) in Assumption 3.4 holds, the Jacobian SDP relaxation [15] is still exact.*

**Corollary 3.11.** *Suppose that*

- (a) *For each subset  $J \subseteq [m_2]$  with  $|J| \leq n - m_1$ ,  $\langle h, g_J \rangle$  is a radical ideal and its codimension is  $m_1 + |J|$ ;*
- (b)  *$V(h)$  is a smooth variety of dimension  $\leq 2$ .*

*Then the condition (iii) in Assumption 3.4 always holds. Therefore, if conditions (i) and (ii) in Assumption 3.4 are satisfied, then the conclusions of Theorem 3.9 hold.*

*Proof.* For any subset  $J \subseteq [m_2]$  with  $|J| \leq n - m_1$ , by (a),  $\Theta_J$  is the set of singularities of  $V(h, g_J)$ . If  $J = \emptyset$ , then  $\Theta_J = \emptyset$  by (b). If  $J \neq \emptyset$ , then by [1, Proposition 3.3.14] or [7, Theorem 5.3],  $\dim \Theta_J < \dim V(h, g_J)$ . Since  $\dim V(h, g_J) \leq 1$  by (a) and (b),  $\Theta_J$  is a finite set for each  $J \subseteq [m_2]$  with  $|J| \leq n - m_1$ . Thus the condition (iii) in Assumption 3.4 always holds.  $\square$

We now give an example to illustrate the finite convergence of the Jacobian SDP relaxation [15] under the weakened Assumption 3.4.

**Example 3.12.** Consider the following polynomial optimization

$$\begin{cases} \min_{x_1, x_2 \in \mathbb{R}} f(x_1, x_2) := x_1 x_2^2 + x_1 \\ \text{s.t. } h(x_1, x_2) := -x_1^3 + x_2^2 = 0. \end{cases}$$

Clearly, the minimum  $f_{\min} = 0$  is achieved at  $(0, 0)$ . However, it is easy to verify that  $(0, 0)$  is a singular point and does not satisfy the KKT conditions. Since  $(0, 0)$  is the only singularity, Assumption 3.4 holds which is also guaranteed by Corollary 3.11. In the following, we show the finite convergence of the Jacobian SDP relaxation [15] by giving the exact equation (17).

By the construction of (13),  $m_1 = 1, m_2 = 0$  and  $r = 1$ .  $\varphi(x_1, x_2) := 2x_2(x_2^2 + 1) + 6x_1^3x_2$ . For any  $\varepsilon > 0$ , let

$$\begin{aligned} \psi(x_1, x_2) := & 8x_1 + 8\varepsilon - 12x_1^8x_2^4 - 24x_1^8x_2^2 + 24x_1^7x_2^2 + 8x_1x_2^2 + 4x_1^3 + 32\varepsilon x_1^3 - \frac{x_1^3}{\varepsilon^2} \\ & + \frac{x_1^4}{8\varepsilon^3} - \frac{2x_1^6}{\varepsilon^2} + \frac{x_1^7}{4\varepsilon^3} + 8\varepsilon x_2^2 + \frac{x_1x_2^2}{64\varepsilon^3} - \frac{x_2^2}{8\varepsilon^2} + \frac{x_1}{64\varepsilon^3} - \frac{1}{8\varepsilon^2} + 4x_1^3x_2^2 \\ & + 4x_1^5x_2^2 + 4x_1^5 + 24x_1^4 - \frac{243x_1^{10}x_2^2}{256\varepsilon^3} - \frac{3x_1^{10}x_2^6}{16\varepsilon^3} - \frac{45x_1^7x_2^4}{128\varepsilon^3} - \frac{311x_1^7x_2^2}{1024\varepsilon^3} \\ & - \frac{3x_1^7x_2^6}{32\varepsilon^3} + \frac{3x_1^3x_2^4}{32\varepsilon^2} + \frac{33x_1^9x_2^2}{8\varepsilon^2} + \frac{29x_1^6x_2^2}{32\varepsilon^2} - \frac{45x_1^4x_2^4}{1024\varepsilon^3} - \frac{45x_1^{10}x_2^4}{64\varepsilon^3} \\ & - \frac{17x_1^3x_2^2}{32\varepsilon^2} + \frac{3x_1^9x_2^4}{2\varepsilon^2} + \frac{3x_1^6x_2^4}{4\varepsilon^2} + \frac{47x_1^4x_2^2}{1024\varepsilon^3} - \frac{3x_1^4x_2^6}{256\varepsilon^3}. \\ \phi(x_1, x_2) := & -\frac{x_1^{10}x_2^5}{32\varepsilon^3} - \frac{15x_1^{10}x_2^3}{128\varepsilon^3} + \frac{x_1^9x_2^3}{4\varepsilon^2} - \frac{x_1^7x_2^5}{64\varepsilon^3} - \frac{81x_1^{10}x_2}{512\varepsilon^3} - 2x_1^8x_2^3 + \frac{11x_1^9x_2}{16\varepsilon^2} \\ & - \frac{15x_1^7x_2^3}{256\varepsilon^3} - 4x_1^8x_2 + \frac{x_1^6x_2^3}{8\varepsilon^2} - \frac{x_1^4x_2^5}{512\varepsilon^3} - \frac{337x_1^7x_2}{2048\varepsilon^3} + 4x_1^7x_2 + \frac{59x_1^6x_2}{64\varepsilon^2} \\ & - \frac{15x_1^4x_2^3}{2048\varepsilon^3} - 2x_1^5x_2 + \frac{x_1^3x_2^3}{64\varepsilon^2} - \frac{x_1^4x_2}{16\varepsilon^3} + \frac{7x_1^3x_2}{16\varepsilon^2} - 2x_1^3x_2 - 4x_1x_2 \\ & - \frac{x_1x_2}{128\varepsilon^3} + \frac{x_2}{16\varepsilon^2} - 4\varepsilon x_2. \end{aligned}$$

$$\sigma_0(x_1, x_2) := 16 \left( \varepsilon + \frac{(x_1 x_2^2 + x_1 + 1)^2}{4} + (x_1 x_2^2 + x_1 - 1)^2 x_2^2 \right) x_1^6 + \varepsilon (4x_1^3 + 1)^2 \left( 1 + \frac{x_1 x_2^2 + x_1}{2\varepsilon} - \frac{(x_1 x_2^2 + x_1)^2}{8\varepsilon^2} \right)^2.$$

It can be verified that

$$f(x_1, x_2) + \varepsilon = \sigma_0(x_1, x_2) + \psi(x_1, x_2)h(x_1, x_2) + \phi(x_1, x_2)\varphi(x_1, x_2).$$

Since each term on the right side of the above equation has degree  $\leq 20$ , we take  $N^* = 10$  in (17). Because  $\sigma_0(x_1, x_2)$  is a sum of squares of polynomials, we have  $\sigma_0(x_1, x_2) \in P^{(10)}$  and  $\psi(x_1, x_2)h(x_1, x_2) + \phi(x_1, x_2)\varphi(x_1, x_2) \in I^{(10)}$ . Therefore,  $f(x_1, x_2) + \varepsilon \in I^{(10)} + P^{(10)}$  for any  $\varepsilon > 0$ . Hence, we have  $f_N^{(1)} = f_N^{(2)} = f_{\min} = 0$  for all  $N \geq 10$ .  $\square$

A practical issue in applications is how to detect whether (15) is exact for a given  $N$ . Nie [15] pointed out that it would be possible to apply the *flat-extension condition* (FEC) [8]. When FEC holds, (15) is exact for (9) and a very nice software *GloptiPoly* [9] provides routines for finding minimizers if FEC holds. In general, the FEC is a sufficient but not necessary condition for checking finite convergence of Lasserre's hierarchy. More recently, Nie [16] proposed the *flat truncation* as a general certificate. For the polynomial optimization (9), define

$$d_{h,i} = \lceil \deg(h_i)/2 \rceil, \quad d_{g,j} = \lceil \deg(g_j)/2 \rceil, \quad d_f = \lceil \deg(f)/2 \rceil, \\ \hat{d} = \max\{1, d_{h,1}, \dots, d_{h,m_1}, d_{g,1}, \dots, d_{g,m_2}\}.$$

When  $\psi \equiv 1$ ,  $L_\psi^{(N)}(y)$  in (14) is called *moment matrix* and is denoted as  $M_N(y) := L_\psi^{(N)}(y)$ . For a given integer  $N \in \mathbb{N}$ , we say an optimizer  $y^*$  of (15) has a flat truncation if there exists an integer  $t \in [\max\{d_f, \hat{d}\}, N]$  such that

$$\text{rank } M_{t-\hat{d}}(y^*) = \text{rank } M_t(y^*).$$

Assuming the set of global minimizers is nonempty and finite, [16, Theorem 2.2 and 2.6] show that the Putinar type or Schmüdgen type Lasserre's hierarchy has finite convergence if and only if the flat truncation holds. As an application, [16, Corollary 4.2] also points out that if (9) has a nonempty set of finitely many global minimizers and Assumption 3.1 is satisfied, then the flat truncation is always satisfied for the hierarchy of Jacobian SDP relaxations. Since we have proved that Assumption 3.1 can be weakened as Assumption 3.4, we have

**Corollary 3.13** (Revised Version of Corollary 4.2 in [16]). *Suppose (9) has a nonempty set of finitely many global minimizers and Assumption 3.4 is satisfied. Then, for all  $N$  big enough, the optimal value of (16) equals the global minimum of (9) and every minimizer of (15) has a flat truncation.*

#### 4. REVISITING MINIMIZATION OF RATIONAL FUNCTIONS

In this section, we return to the minimization of (1). We first apply the Jacobian SDP relaxation discussed in Section 3 to reformulate (3) as (13) for which we consider the finite convergence of the SDP relaxations. Next, we do some numerical experiments to show the efficiency of our method.

**4.1. Minimizing Rational Functions by Jacobian SDP Relaxation.** Consider the number of new constraints added when we employ Jacobian SDP relaxation to solve (3). As mentioned in [15], the number of new constraints in (13) is exponential in the number of inequality constraints. Hence, if the number of inequality constraints is large, (13) becomes more difficult to solve numerically. In the following, we employ the Jacobian SDP relaxation to reformulate (3) as (13). We show that the number of the new equality constraints  $\varphi_i$ 's in (13) can be reduced due to the special inequality constraint  $x_0 \geq 0$  in (3).

In (3), for convenience, let

$$h_{m_1+1}^{hom}(\tilde{x}) := \tilde{q}(\tilde{x}) - 1 = 0, \quad g_{m_2+1}^{hom}(\tilde{x}) := x_0 \geq 0 \quad \text{and} \quad m := \min\{m_1 + m_2 + 2, n\}.$$

Denote

$$\nabla_{\tilde{x}} := \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

According to (10) and (11), we need to consider all subsets of  $[m_2 + 1]$  with cardinality  $\leq m - m_1 - 1$ . Let  $l = \min\{m - m_1 - 1, m_2\}$ . We first consider the subsets without  $m_2 + 1$ , i.e., every subset  $J = \{j_1, \dots, j_k\} \subseteq [m_2]$  with  $k \leq l$ . Denote  $h^{hom} = (h_1^{hom}, \dots, h_{m_1}^{hom}, h_{m_1+1}^{hom})$  and  $g_J^{hom} = (g_{j_1}^{hom}, \dots, g_{j_k}^{hom})$ . Let  $\{\eta_1, \dots, \eta_{len(J)}\}$  be the set of the defining equations for the determinantal variety

$$G_J := \{\tilde{x} \in \mathbb{C}^{n+1} \mid \text{rank} [\nabla_{\tilde{x}} \tilde{p} \quad \nabla_{\tilde{x}} h^{hom} \quad \nabla_{\tilde{x}} g_J^{hom}] \leq m_1 + |J| + 1\}.$$

For each  $i = 1, \dots, len(J)$ , define

$$\varphi_i^J(\tilde{x}) = \eta_i \cdot \prod_{j \in J^c} g_j^{hom}(\tilde{x}), \quad \text{where} \quad J^c = [m_2 + 1] \setminus J.$$

For every subset  $J$  considered above, denote  $J' = J \cup \{m_2 + 1\} \subseteq [m_2 + 1]$ . It can be checked that the collection of these  $J$ 's and  $J'$ 's contains all subsets of  $[m_2 + 1]$  with cardinality  $\leq m - m_1 - 1$  and some possible  $J'$ 's with cardinality  $= m - m_1$  (which will happen when  $n < m_1 + m_2 + 2$ ).

**Case  $|J'| \leq m - m_1 - 1$**  All these  $J'$ 's compose of the subsets of  $[m_2 + 1]$  containing  $m_2 + 1$  with cardinality  $\leq m - m_1 - 1$ . It is easy to see that the set of the defining equations for the determinantal variety

$$G_{J'} := \{\tilde{x} \in \mathbb{C}^{n+1} \mid \text{rank} [\nabla_{\tilde{x}} \tilde{p} \quad \nabla_{\tilde{x}} h^{hom} \quad \nabla_{\tilde{x}} g_{J'}^{hom} \quad \nabla_{\tilde{x}} x_0] \leq m_1 + |J| + 2\}$$

is a subset of  $\{\eta_1, \dots, \eta_{len(J)}\}$ . We generally suppose it to be  $\{\eta_1, \dots, \eta_{t(J)}\}$  with  $t(J) < len(J)$ . For  $i = 1, \dots, t(J)$ , define

$$\varphi_i^{J'}(\tilde{x}) = \eta_i \cdot \prod_{j \in J'^c} g_j^{hom}(\tilde{x}), \quad \text{where} \quad J'^c = [m_2 + 1] \setminus J'.$$

**Case  $|J'| = m - m_1$**  It is easy to check that  $G_{J'} = \mathbb{C}^{n+1}$ . Thus for convenience, we set  $t(J) = 0$  in this case.

Then for every subset  $J \subseteq [m_2]$  with  $|J| \leq l$ , we have

$$(18) \quad \varphi_i^J(\tilde{x}) = \varphi_i^{J'}(\tilde{x}) \cdot x_0, \quad i = 1, \dots, t(J).$$



Now consider the SDP relaxations [13] for the following polynomial optimization

$$(19) \quad \begin{cases} p^* := \min_{\tilde{x} \in \mathbb{R}^{n+1}} \tilde{p}(\tilde{x}) \\ \text{s.t. } h_1^{hom}(\tilde{x}) = \dots = h_{m_1}^{hom}(\tilde{x}) = h_{m_1+1}^{hom}(\tilde{x}) = 0, \\ \varphi_i^J(\tilde{x}) = 0, \varphi_j^{J'}(\tilde{x}) = 0 \\ (i \in [len(J)], j \in [t(J)], J \subseteq [m_2], |J| \leq l), \\ g_\nu^{hom}(\tilde{x}) \geq 0, \forall \nu \in \{0, 1\}^{m_2+1}, \end{cases}$$

where  $g_\nu^{hom} = (g_1^{hom})^{\nu_1} \dots (g_{m_2+1}^{hom})^{\nu_{m_2+1}}$ . We now show that for each  $J \subseteq [m_2]$  with  $|J| \leq l$ , constraints  $\varphi_1^J(\tilde{x}) = \dots = \varphi_{t(J)}^J(\tilde{x}) = 0$  can be removed from (19). Consider the  $N$ -th order SDP relaxation (15) for (19). By (18) and the properties of localizing moment matrices in [14, Lemma 4.1], we have

$$L_{\varphi_j^{J'}}^{(N)}(y) = 0 \quad \text{implies} \quad L_{\varphi_j^J}^{(N)}(y) = 0, \quad j = 1, \dots, t(J), \quad J \subseteq [m_2], \quad |J| \leq l.$$

In the dual problem (16), by (18), the truncated ideal

$$\left\{ \sum_{J \subseteq [m_2], |J| \leq l} \left( \sum_{i=1}^{len(J)} \phi_i \varphi_i^J + \sum_{j=1}^{t(J)} \zeta_j \varphi_j^{J'} \right) + \sum_{k=1}^{m_1+1} \psi_k h_k^{hom} \right\}, \text{ where}$$

$$\forall i, j, k, \deg(\phi_i \varphi_i^J) \leq 2N, \deg(\zeta_j \varphi_j^{J'}) \leq 2N, \deg(\psi_k h_k^{hom}) \leq 2N,$$

agrees with

$$(20) \quad \left\{ \sum_{J \subseteq [m_2], |J| \leq l} \left( \sum_{i=t(J)+1}^{len(J)} \phi_i \varphi_i^J + \sum_{j=1}^{t(J)} \zeta_j \varphi_j^{J'} \right) + \sum_{k=1}^{m_1+1} \psi_k h_k^{hom} \right\} \text{ where}$$

$$\forall i, j, k, \deg(\phi_i \varphi_i^J) \leq 2N, \deg(\zeta_j \varphi_j^{J'}) \leq 2N, \deg(\psi_k h_k^{hom}) \leq 2N.$$

Therefore, we can remove  $\varphi_1^J(\tilde{x}) = \dots = \varphi_{t(J)}^J(\tilde{x}) = 0$  in (19) and improve the numerical performance in practice. Hence we consider the following optimization

$$(21) \quad \begin{cases} p^* := \min_{\tilde{x} \in \mathbb{R}^{n+1}} \tilde{p}(\tilde{x}) \\ \text{s.t. } h_1^{hom}(\tilde{x}) = \dots = h_{m_1}^{hom}(\tilde{x}) = h_{m_1+1}^{hom}(\tilde{x}) = 0, \\ \varphi_i^J(\tilde{x}) = 0, \varphi_j^{J'}(\tilde{x}) = 0 \\ (i = t(J) + 1, \dots, len(J), j \in [t(J)], J \subseteq [m_2], |J| \leq l), \\ g_\nu^{hom}(\tilde{x}) \geq 0, \forall \nu \in \{0, 1\}^{m_2+1}, \end{cases}$$

The  $N$ -th order SDP relaxation [13] for (21) is the SDP

$$(22) \quad \begin{cases} p_N^{(1)} := \min L_{\tilde{p}}(y) \\ \text{s.t. } L_{h_1^{hom}}^{(N)}(y) = \dots = L_{h_{m_1+1}^{hom}}^{(N)}(y) = L_{h_{m_1+1}^{hom}}^{(N)}(y) = 0, \\ L_{\varphi_i^J}^{(N)}(y) = 0, L_{\varphi_j^{J'}}^{(N)}(y) = 0 \\ (i = t(J) + 1, \dots, len(J), j \in [t(J)], J \subseteq [m_2], |J| \leq l), \\ L_{g_\nu^{hom}}^{(N)} \succeq 0, \forall \nu \in \{0, 1\}^{m_2+1}, y_0 = 1. \end{cases}$$

The dual problem of (22) is

$$(23) \quad p_N^{(2)} := \max_{\gamma \in \mathbb{R}^{n+1}} \gamma \quad \text{s.t.} \quad \tilde{p}(\tilde{x}) - \gamma \in I^{(N)} + P^{(N)},$$

where  $I^{(N)}$  is the ideal defined in (20) and

$$P^{(N)} = \left\{ \sum_{\nu \in \{0,1\}^{m_2+1}} \sigma_\nu g_\nu^{hom} \mid \deg(\sigma_\nu g_\nu^{hom}) \leq 2N \right. \\ \left. \sigma_\nu \text{'s are SOS} \right\}.$$

**Definition 4.1.** For every set  $J = \{j_1, \dots, j_k\} \subseteq [m_2 + 1]$  with  $k \leq n - m_1$ , let

$$\Theta_J = \{\tilde{x} \in V(h^{hom}, g_J^{hom}) \mid \text{rank} \begin{bmatrix} \nabla_{\tilde{x}} h^{hom} & \nabla_{\tilde{x}} g_J^{hom} \end{bmatrix} < m_1 + |J| + 1\}$$

and

$$\Theta = \bigcup_{J \subseteq [m_2+1], |J| \leq n-m_1} \Theta_J.$$

**Assumption 4.2.** (i)  $m_1 \leq n$ ; (ii) For any  $u \in \tilde{S}$  in (5), at most  $n - m_1$  of  $g_1^{hom}(u), \dots, g_{m_2+1}^{hom}(u)$  vanish; (iii) The set  $\Theta$  is finite.

By Theorem 2.3 and 3.9, we have

**Theorem 4.3.** Suppose Assumption 4.2 holds. Then  $p^* > -\infty$  in (21) and there exists  $N^* \in \mathbb{N}$  such that  $p_N^{(1)} = p_N^{(2)} = p^*$  for all  $N \geq N^*$ . Furthermore, if one of the conditions in Theorem 2.3 holds and the minimum  $s^*$  of (3) is achievable, then  $p_N^{(1)} = p_N^{(2)} = r^*$  for all  $N \geq N^*$ .

**Corollary 4.4.** If  $S = \mathbb{R}^n$  in (1) and  $s^*$  is achievable in (7), then there exists  $N^* \in \mathbb{N}$  such that  $p_N^{(1)} = p_N^{(2)} = r^*$  for all  $N \geq N^*$  in (22) and (23).

*Proof.* Since  $\tilde{q}$  is homogeneous, regarding  $\nabla \tilde{q}$  and  $\tilde{x}$  as vectors in  $\mathbb{R}^{n+1}$ , then  $d \cdot \tilde{q} = \nabla \tilde{q}^T \cdot \tilde{x}$  by Euler's Formula. Thus  $\nabla(\tilde{q} - 1) = \nabla \tilde{q} = 0$  implies  $\tilde{q} = 0$ , i.e.  $\Theta = \emptyset$ . Hence, Assumption 4.2 is always true for (7). Then by Corollary 2.4 and Theorem 4.3, the conclusion follows.  $\square$

In the end of this subsection, we would like to point out that  $s^*$  in (3) might not be achievable in some cases. If the infimum of a constrained polynomial optimization is asymptotic value, some approaches are proposed in [5, 29]. Hence, we can still use these approaches to solve (3). However, to the best knowledge of the authors, the finite convergence for these methods is unknown.

**4.2. Numerical Experiments.** In this subsection, we present some numerical examples to illustrate the efficiency of our method for solving minimization of (1). We use the software *GloptiPoly* [9] to solve (22) and (23).

**4.2.1. Unconstrained rational optimization.** In the following, Example 4.5 and 4.6 are constructed from the *Motzkin* polynomial

$$(24) \quad M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2.$$

As is well-known,  $M(x_1, x_2, x_3)$  is nonnegative on  $\mathbb{R}^3$  but not SOS [25].

**Example 4.5.** [18, Example 2.9] Consider minimization

$$(25) \quad \min_{x_1, x_2 \in \mathbb{R}} r(x_1, x_2) := \frac{x_1^4 x_2^2 + x_1^2 x_2^4 + 1}{x_1^2 x_2^2}.$$

Taking  $x_3 = 1$  in Motzkin polynomial, we have  $x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - 3x_1^2 x_2^2 \geq 0$  on  $\mathbb{R}^2$ . Since  $r(1, 1) = 3$ , we have  $r^* = 3$  and there are four minimizers  $(\pm 1, \pm 1)$ . However,  $x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - r^* x_1^2 x_2^2$  is not SOS. To solve this problem, the authors in [18] used the generalized big ball technique. More specifically, it is assumed that one of the minimizers of (25) lies in a ball  $B(c, \rho)$  and the numerator and denominator of  $r(x_1, x_2)$  have no common real roots on  $B(c, \rho)$ . However, it is not easy in general to determine the radius  $\rho$  of this ball. We now solve this problem using our method without the assumptions in [18].

We first reformulate the problem as the following polynomial optimization:

$$\begin{cases} \min_{x_0, x_1, x_2 \in \mathbb{R}} \tilde{p}(x_0, x_1, x_2) := x_1^4 x_2^2 + x_1^2 x_2^4 + x_0^6 \\ \text{s.t. } \tilde{q}(x_0, x_1, x_2) := x_1^2 x_2^2 x_0^2 = 1. \end{cases}$$

By Jacobian SDP relaxation (21), we need 3 more equations:

$$\begin{aligned} \varphi_1(x_0, x_1, x_2) &= 4x_1^3 x_2^3 x_0^2 (x_1^2 - x_2^2) = 0 \\ \varphi_2(x_0, x_1, x_2) &= 4x_1 x_2^2 x_0 (2x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_0^6) = 0 \\ \varphi_3(x_0, x_1, x_2) &= 4x_1^2 x_2 x_0 (x_1^4 x_2^2 + 2x_1^2 x_2^4 - 3x_0^6) = 0 \end{aligned}$$

By the condition  $x_1^2 x_2^2 x_0^2 = 1$ , the above three equations can be simplified as

$$\begin{aligned} \varphi_1(x_0, x_1, x_2) &= x_1^2 - x_2^2 = 0 \\ \varphi_2(x_0, x_1, x_2) &= 2x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_0^6 = 0 \\ \varphi_3(x_0, x_1, x_2) &= x_1^4 x_2^2 + 2x_1^2 x_2^4 - 3x_0^6 = 0 \end{aligned}$$

We need to solve the following new problem

$$\begin{cases} \min_{x_0, x_1, x_2 \in \mathbb{R}} x_1^4 x_2^2 + x_1^2 x_2^4 + x_0^6 \\ \text{s.t. } x_1^2 x_2^2 x_0^2 - 1 = 0, \quad 2x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_0^6 = 0, \\ \quad x_1^2 - x_2^2 = 0, \quad x_1^4 x_2^2 + 2x_1^2 x_2^4 - 3x_0^6 = 0. \end{cases}$$

Using GloptiPoly to solve this problem, we get the following results:

- $N = 3$ . The optimum is 3, but extracting global optimal solutions fails.
- $N = 4$ . We get 8 optimal solutions for  $(x_0, x_1, x_2)$ :  $(\pm 1, \pm 1, \pm 1)$  from which we get all the optimal solutions for original problem:  $(\pm 1, \pm 1)$ .  $\square$

**Example 4.6.** [18, Example 2.10] Consider the following problem

$$(26) \quad \min_{x_1, x_2 \in \mathbb{R}} r(x_1, x_2) := \frac{p(x_1, x_2)}{q(x_1, x_2)} = \frac{x_1^4 + x_1^2 + x_2^6}{x_1^2 x_2^2}.$$

Taking  $x_2 = 1$  in Motzkin polynomial (24), we have  $r^* = 3$  with 4 minimizers  $(\pm 1, \pm 1)$ . The denominator and numerator have real common root  $(0, 0)$ . In [18], the SOS relaxation extracts 6 solutions, 2 of which are not global minimizers but

the common roots of  $p(x)$  and  $q(x)$ . We reformulate it as the following polynomial optimization and solve it by Jacobian SDP relaxation.

$$(27) \quad \begin{cases} \min_{x_0, x_1, x_2 \in \mathbb{R}} & \tilde{p}(x_0, x_1, x_2) := x_1^4 x_0^2 + x_1^2 x_0^4 + x_2^6 \\ \text{s.t.} & \tilde{q}(x_0, x_1, x_2) := x_1^2 x_2^2 x_0^2 = 1. \end{cases}$$

Using GloptiPoly, we can still extract 8 solutions of (27) and obtain all the 4 optimal solutions of (26) as in Example 4.5. In our method, the constraint  $\tilde{q}(x_0, x_1, x_2) = 1$  prevents extracting the common real roots of  $p(x)$  and  $q(x)$ . This example also shows that Condition (b) in Theorem 2.7 is only sufficient but not necessary.  $\square$

The following example is generated from the *Robinson* polynomial

$$x_1^6 + x_2^6 + x_3^6 - (x_1^4 x_2^2 + x_1^2 x_2^4 + x_1^4 x_3^2 + x_1^2 x_3^4 + x_2^4 x_3^2 + x_2^2 x_3^4) + 3x_1^2 x_2^2 x_3^2$$

which is nonnegative on  $\mathbb{R}^3$  but not SOS [25].

**Example 4.7.** Consider the following problem

$$(28) \quad \min_{x_1, x_2 \in \mathbb{R}} r(x_1, x_2) := \frac{p(x_1, x_2)}{q(x_1, x_2)} = \frac{x_1^6 + x_2^6 + 3x_1^2 x_2^2 + 1}{x_1^2(x_2^4 + 1) + x_2^2(x_1^4 + 1) + (x_1^4 + x_2^4)}.$$

Taking  $x_3 = 1$  in Robinson polynomial, we have  $p(x_1, x_2) - q(x_1, x_2) \geq 0$  on  $\mathbb{R}^2$ . Since  $r(1, 1) = 1$ ,  $r^* = 1$ . We reformulate it as the following optimization:

$$(29) \quad \begin{cases} \min_{x_0, x_1, x_2 \in \mathbb{R}} & x_1^6 + x_2^6 + 3x_1^2 x_2^2 x_0^2 + x_0^6 \\ \text{s.t.} & x_1^2(x_2^4 + x_0^4) + x_2^2(x_1^4 + x_0^4) + x_0^2(x_1^4 + x_2^4) = 1. \end{cases}$$

The numerical results we obtained are:

- For relaxation order  $N = 5, 6$ , we get the optimum  $s^* = 1$ , but the minimizers can not be extracted.
- For relaxation order  $N = 7$ , we extract 20 approximate minimizers of (29):  
 $(-0.0000, \pm 0.8909, \pm 0.8909), \quad (\pm 0.8909, \pm 0.8909, -0.0000),$   
 $(\pm 0.8909, -0.0000, \pm 0.8909), \quad (\pm 0.7418, \pm 0.7418, \pm 0.7418).$

The above solutions correspond to the exact minimizers of (29):

$$\begin{aligned} & (0, \pm \frac{1}{\sqrt[6]{2}}, \pm \frac{1}{\sqrt[6]{2}}), \quad (\pm \frac{1}{\sqrt[6]{2}}, \pm \frac{1}{\sqrt[6]{2}}, 0), \\ & (\pm \frac{1}{\sqrt[6]{2}}, 0, \pm \frac{1}{\sqrt[6]{2}}), \quad (\pm \frac{1}{\sqrt[6]{6}}, \pm \frac{1}{\sqrt[6]{6}}, \pm \frac{1}{\sqrt[6]{6}}). \end{aligned}$$

There are four solutions with the first coordinate  $x_0^* = 0$  which indicate that minimum  $r^* = 1$  is also an asymptotic value at  $\infty$  by Theorem 2.7. In fact,

$$\lim_{x_1, x_2 \rightarrow \infty} \frac{p(x_1, x_2)}{q(x_1, x_2)} = 1 = r^*.$$

From the other 16 solutions, according to (a) in Theorem 2.7, we get 8 global minimizers of (28):  $(\pm 1, \pm 1), (\pm 1, 0), (0, \pm 1)$ .  $\square$

**Example 4.8.** [18, Example 3.4] Suppose function  $\psi(z)$  and  $\phi(z)$  are monic complex univariate polynomials of degree  $m$  such that:

$$\begin{aligned} \psi(z) &= z^m + \psi_{m-1}z^{m-1} + \cdots + \psi_1z + \psi_0 \\ \phi(z) &= z^m + \phi_{m-1}z^{m-1} + \cdots + \phi_1z + \phi_0 \end{aligned}$$

It is shown in [11] that finding nearest GCDs becomes the following global minimization of rational functions

$$(30) \quad \min_{x_1, x_2 \in \mathbb{R}} \frac{p(x_1, x_2)}{q(x_1, x_2)} = \frac{|\psi(x_1 + ix_2)|^2 + |\phi(x_1 + ix_2)|^2}{\sum_{k=0}^{m-1} (x_1^2 + x_2^2)^k}$$

where  $\deg(p) = 2m$  and  $\deg(q) = 2(m-1)$ . Let

$$\psi(z) = z^3 + z^2 - 2, \quad \phi(z) = z^3 + 1.5z^2 + 1.5z - 1.25.$$

Using our method for relaxation order  $N = 5$ , we get four optimal solutions of the optimization reformulated from (30) by homogenization:

$$(0.7050, -0.7073, \pm 0.7763), \quad (-0.7050, 0.7073, \pm 0.7763).$$

The corresponding minimizers of (30) are

$$(x_1 \approx -1.0033, x_2 \approx \pm 1.1011), \quad (x_1 \approx -1.0033, x_2 \approx \pm 1.1011)$$

which are the same as in [18]. The minimum is  $r^* \approx 0.0643$ .  $\square$

**4.2.2. Constrained rational optimization.** We now give some numerical examples of minimizing of rational functions with polynomial inequality constraints. We first consider an example for which  $p(x)$  and  $q(x)$  have common roots.

**Example 4.9.** [18] Consider the following optimization

$$(31) \quad \min_{x \in \mathbb{R}} r(x) := \frac{1+x}{(1-x^2)^2} \quad \text{s.t. } (1-x^2)^3 \geq 0.$$

As shown in [18], the global minimum  $r^* = \frac{27}{32} \approx 0.8438$  and the minimizer  $x^* = -\frac{1}{3} \approx -0.3333$ . If the denominator and numerator have common roots, SOS relaxation method proposed in [18] can not guarantee to converge to the minimum.

Reformulating the above problem by homogenization, we get

$$(32) \quad \begin{cases} \min_{x_0, x_1 \in \mathbb{R}} x_0^4 + x_1 x_0^3 \\ \text{s.t. } x_0^4 - 2x_1^2 x_0^2 + x_1^4 = 1, \\ x_0^6 - 3x_0^4 x_1^2 + 3x_0^2 x_1^4 - x_1^6 \geq 0, \quad x_0 \geq 0. \end{cases}$$

For relaxation order  $N = 7$ , by the Jacobian SDP relaxation, we get the optimal solution of (32)  $\tilde{x}^* \approx (1.0607, -0.3536)$  and the minimum  $s^* \approx 0.8437$ . According to (a) in Theorem 2.7, we find the minimizer of (31):  $x^* \approx -0.3334$ .  $\square$

We next consider Example 4.7 with some constraints.

**Example 4.10.** Consider optimization

$$(33) \quad r^* := \min_{x \in S} \frac{p(x_1, x_2)}{q(x_1, x_2)} = \frac{x_1^6 + x_2^6 + 3x_1^2 x_2^2 + 1}{x_1^2(x_2^4 + 1) + x_2^2(x_1^4 + 1) + (x_1^4 + x_2^4)}.$$

(a)  $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ . It is easy to check that  $S$  is closed at  $\infty$ .

By Theorem 2.3,  $r^*$  is equal to the optimum of the following optimization:

$$\begin{cases} s^* = \min_{x_0, x_1, x_2 \in \mathbb{R}} x_1^6 + x_2^6 + 3x_1^2 x_2^2 x_0^2 + x_0^6 \\ \text{s.t. } x_1^2(x_2^4 + x_0^4) + x_2^2(x_1^4 + x_0^4) + x_0^2(x_1^4 + x_2^4) = 1, \\ x_0^2 - x_1^2 - x_2^2 \geq 0, \quad x_0 \geq 0. \end{cases}$$

For relaxation order  $N = 7$ , we get  $r^* = s^* = 1$  with 4 approximate minimizers:

$$(0.8909, \pm 0.8909, -0.0000), \quad (0.8909, -0.0000, \pm 0.8909),$$

which correspond to the exact minimizers:

$$\left(\frac{1}{\sqrt[6]{2}}, \pm \frac{1}{\sqrt[6]{2}}, 0\right), \quad \left(\frac{1}{\sqrt[6]{2}}, 0, \pm \frac{1}{\sqrt[6]{2}}\right).$$

Then we get four minimizers of (33):  $(\pm 1, 0)$ ,  $(0, \pm 1)$ .

- (b)  $S = B(0, \sqrt{2})^c = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 2\}$ .  $S$  is noncompact but closed at  $\infty$ . By Theorem 2.3, we solve the following equivalent optimization:

$$\begin{cases} s^* := \min_{x_0, x_1, x_2 \in \mathbb{R}} x_1^6 + x_2^6 + 3x_1^2 x_2^2 x_0^2 + x_0^6 \\ \text{s.t. } x_1^2(x_2^4 + x_0^4) + x_2^2(x_1^4 + x_0^4) + x_0^2(x_1^4 + x_2^4) = 1, \\ x_1^2 + x_2^2 - 2x_0^2 \geq 0, \quad x_0 \geq 0. \end{cases}$$

For relaxation order  $N = 7$ , we get  $r^* = s^* = 1$  with 8 approximate minimizers:

$$(0.0002, \pm 0.8909, \pm 0.8909), \quad (0.7418, \pm 0.7419, \pm 0.7419),$$

which correspond to the exact minimizers:

$$\left(0, \pm \frac{1}{\sqrt[6]{2}}, \pm \frac{1}{\sqrt[6]{2}}\right), \quad \left(\frac{1}{\sqrt[6]{6}}, \pm \frac{1}{\sqrt[6]{6}}, \pm \frac{1}{\sqrt[6]{6}}\right).$$

The former solutions indicate that  $r^* = 1$  is also an asymptotic values at  $\infty$ . From the latter solutions, we get four minimizers of (33):  $(\pm 1, \pm 1)$ .  $\square$

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